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2001 J. Phys.: Condens. Matter 13 8031

(http://iopscience.iop.org/0953-8984/13/35/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **13** (2001) 8031–8038 PII: S0953-8984(01)26042-1

Collective behaviour in electrical dipolar systems

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Received 22 June 2001
Published 16 August 2001
Online at stacks.iop.org/JPhysCM/13/8031

Abstract

This paper reports a new wave phenomenon. We show that in a system consisting of many electric dipoles, the interaction between dipoles, mediated by radiated electromagnetic waves, leads to a kind of phase transition in the system. When the phase transition occurs, all dipoles tend to oscillate in phase, displaying a global coherent behaviour. It is also found that this feature is independent of the precise configuration of the dipole system.

1. Introduction

One of the greatest advances in the last century is the achieving of an understanding of how a system having many interacting elements is manifested and behaves in the presence of mutual interactions, giving rise to many important phenomena including attenuation of both classical and quantum waves, the quantum Hall effect, superconductivity, superfluidity, plasma excitation, and so on [1,2]. The interaction between the constituent bodies not only causes the properties of the individual entities to change, but under appropriate conditions makes possible the appearance of a global collective behaviour in the system, thereby provoking a transition of the system from one state to another. For example, consider a normal metal. The exchange of phonons between electrons in the metal introduces an attractive force. For sufficiently low temperatures, this force can overcome the Coulomb repulsion for a range of frequencies, and thus it becomes a binding force by means of which the electrons in the metal form bound pairs, a picture first described by Cooper [3]. The pair state is the macroscopic superconducting state, and in this state the resistivity disappears. In this way, the metal undergoes a transition from the normal state with non-zero resistivity to a superconducting state. In the superconducting state, the global collective behaviour that has emerged is reflected by, for example, the uniform electrical potential across the system [4]. It has become well known that transitions between different macroscopic states originate from interactions between individuals in the systems.

The mutual-interaction-induced phase transition in a system consisting of many constituents has been very much studied in the context of quantum situations [1]. It is now a well-known fact that in quantum systems, although there are various forms, such a transition is often due to the wave nature of the constituent entities. It is the multiple interaction and interference of the waves that lead to the transition. By analogy, this led to the conjecture that similar phase transitions may also occur in systems involving classical waves such as acoustic and electromagnetic waves. Indeed, it was recently pointed out [5] that there is a phase transition in the propagation of acoustic waves in water with many parallel air-filled cylinders. It was shown that when the number density of the cylinders exceeds a certain amount, the propagating wave comes to a complete halt, over a range of acoustic frequencies. Meanwhile, all cylinders will oscillate completely in phase, exhibiting a globally collective behaviour in the system. In this paper, we consider another classical system: the system consisting of many electric dipoles. We show that under appropriate conditions, the interaction between the electric dipoles, mediated by the radiated electrical fields, can lead to a previously unreported coherent behaviour; that is, over a range of frequencies a new phase state appears. In the new phase state, a global collective behaviour arises: namely, all dipoles tend to oscillate completely in phase, very much in the same way as in the oscillation of air cylinders in water in response to the incidence of proper acoustic waves [5].

2. Model

Consider a system of N dipoles. The dipoles are distributed either randomly or periodically in the space. Their coordinates are $\vec{r}_i$ with i running from 1 to N. The conceptual layout of the system is illustrated in figure 1. For brevity, we assume that the centres of these dipoles are fixed in the space, an assumption valid as long as the motion of the dipoles is slow compared to the speed of light. Although our formulation includes the case where all dipoles point in various directions, here for simplicity we only consider the case where all dipoles point in one direction, assumed to be the z-direction. Further, the rotation of the dipoles is neglected. In the presence of any stimulation, the dipoles will oscillate and subsequently radiate waves. The radiated waves will again excite the oscillation of other dipoles. Such a process will be repeated to establish an infinite recursive pattern of re-excitation and re-radiation. Obviously, the interaction between dipoles is mediated by the radiated waves. Such a mutual interaction can be studied using a set of coupled equations.

When a dipole $\vec{p}$ oscillates, the electrical wave will be radiated. The radiated field at the spatial point $\vec{r}$ is given as

$$\vec{E}(\vec{r},t) = \frac{\mu_0}{4\pi r}[\hat{r} \times (\hat{r} \times \vec{p}(t'))]\big|_{t'=t-r/c} + \vec{E}_n(\vec{r},t) \tag{1}$$

where the near field is

$$\vec{E}_n(t,t) = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{r^3}\hat{r}(\hat{r}\cdot\vec{p}) - \frac{\vec{p}}{r^3} + \frac{3}{r^2 c}\hat{r}(\hat{r}\cdot\ddot{\vec{p}}) - \frac{\ddot{\vec{p}}}{r^2 c}\right)$$

and c is the phase speed of the electrical wave, and $\hat{r}$ is the unit vector along the direction of $\vec{r}$. The second term on the right-hand side of equation (1) results from the near-field effect. The dipoles interact with each other through the radiated electrical waves. Consider, for instance, the ith dipole; its equation of motion can be derived as [6]

$$\frac{\mathrm{d}^2 Q_i}{\mathrm{d}t^2} + \omega_i^2 Q_i + \gamma_i \frac{\mathrm{d}Q_i}{\mathrm{d}t} = -F_i(t) + \sum_{j=1,j\neq i}^{N} C_{ij}\left(1 - \frac{1}{2}Q_i \hat{z}_i \cdot \vec{\nabla}\right)\frac{\mathrm{D}^{ij}}{\mathrm{D}t^2}\frac{Q_j(t-|\vec{r}_i-\vec{r}_j|/c)}{|\vec{r}_i-\vec{r}_j|} \tag{2}$$

where F_i is an external stimulus, and the operator $\mathrm{D}^{ij}/\mathrm{D}t^2$ is given as

$$\frac{\mathrm{D}^{ij}}{\mathrm{D}t^2} = z_{ij}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{3c}{|\vec{r}_i-\vec{r}_j|}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{3c^2}{|\vec{r}_i-\vec{r}_j|^2}\right) + 2\hat{z}_i\cdot\hat{z}_j\left(\frac{c}{|\vec{r}_i-\vec{r}_j|}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{c^2}{|\vec{r}_i-\vec{r}_j|^2}\right)$$

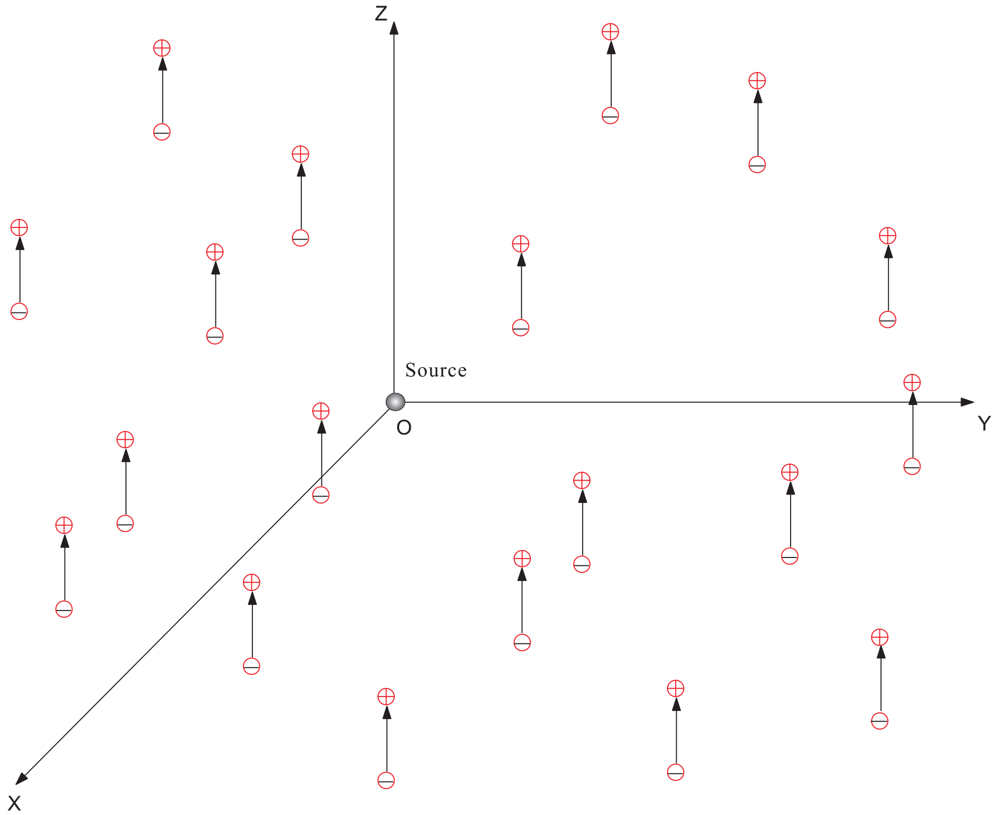

Figure 1. The conceptual layout of a many-dipole system.

with

$$z_{ij} \equiv [(\hat{r}_{ij} \cdot \hat{z}_i)(\hat{r}_{ij} \cdot \hat{z}_j) - \hat{z}_i \cdot \hat{z}_j].$$

Here we ignore the static fields, as they only modify the resonance of the dipoles and can be scaled into the natural frequency. We also make the approximation that the oscillation amplitude is of the order of the dipole size. In equation (2), Q_i is the oscillation displacement of the dipole with ω_i being the natural frequency of the dipole and γ_i being the damping factor, $\hat{z}_i$ is the unit vector along the dipole direction of the ith dipole, $\hat{r}_{ij}$ is the unit vector pointing from the jth dipole to the ith dipole, and the coupling constant $C_{ij} = q_i q_j \mu_0/(2\pi m_i)$ with q_i and m_i being the charge and mass of the dipole respectively. The dipoles and the radiated waves are the system considered in this paper. Note that the damping effect can be caused by various mechanisms, including ones such as the radiation and friction damping. The radiation mechanism only converts the oscillation energy of the dipoles into energy of radiated fields without energy loss. The radiation damping can be estimated as $\gamma_i^{rad} = \mu_0 q_i^2 \omega^2/(12\pi c m_i)$. In contrast, dissipative mechanisms like the friction damping will convert the energy of the system into heat, and therefore are the agents of energy loss. In the model presented by equation (2), the factor γ_i can be adjusted to simulate situations with or without dissipative mechanisms.

The physical meaning of equation (2) is clear. The left-hand side refers to the bare motion of the isolated dipole. The presence of both the external stimulation and other dipoles contributes to the force on the right-hand side, driving the dipole to oscillate. The second term in the summation on the right-hand side of equation (2) is the non-linear interaction between dipoles. In the present paper, we assume that the oscillation amplitude of the dipoles is small so that the non-linear interaction can be neglected. To explore the properties of the system in an explicit way, we make a further reasonable simplification. We assume that all electric dipoles are identical, having the same damping rate, the same mass and the same natural frequencies. There is an arbitrary point source located inside the dipole cloud.

As a result of the above simplification, the governing equation for any dipole is reduced to

$$\frac{\mathrm{d}^2 Q_i}{\mathrm{d}t^2} + \omega_0^2 Q_i + \gamma \frac{\mathrm{d}Q_i}{\mathrm{d}t} = -F_i(t) + \sum_{j=1, j\neq i}^{N} \frac{C}{|\vec{r}_i - \vec{r}_j|} \frac{\mathrm{D}^{ij}}{\mathrm{D}t^2} Q_j(t - |\vec{r}_i - \vec{r}_j|/c). \tag{3}$$

Equation (3) can be solved rigorously by numerical computation. In particular, we have considered two random distributions of the dipoles. One is three dimensional. The number density of the dipoles is n; the average distance between the dipoles is thus $d = n^{-1/3}$. The dipoles form a spherical cloud, i.e. the dipoles are located randomly inside a sphere with radius $R = (3N/[4\pi n])^{1/3}$. We define a size a such that as long as the separation between dipoles is greater than a, equation (1) holds. Clearly, a can be regarded as a measure of the range of a single dipole. The volume fraction covered by the dipoles is subsequently defined as $\beta = N(a/R)^3$, which is a non-dimensional parameter and will be used in the computation. With a fixed β, the average separation between dipoles is calculated as $d/a = (4\pi/[3\beta])^{1/3}$, and the radius of the dipole cloud is measured by $R/a = (N/\beta)^{1/3}$. The other random distribution is two dimensional. We put all dipoles on the x–y plane, with all dipoles pointing in the positive z-direction. We can also define a non-dimensional parameter representing the density of the dipoles. Assume the range covered by a single dipole is of the order of a size of a. The area density of the dipoles is n. Define β as the area fraction taken by the dipoles, that is, $\beta = n(\pi a^2)$. The average distance between dipoles is then computed as $d/a = (\pi/\beta)^{1/2}$. The dipoles are enclosed inside a circle with radius $R/a = (N/\beta)^{1/2}$. For both cases, a stimulating point source is placed at the centre of the dipole clouds; thus $F_i(t) = f(t - r_i/c)/r_i$.

3. Numerical results and discussion

We numerically evaluate equation (3) in the frequency domain. Applying the Fourier transformation to both sides of the equation, we obtain

$$\left(-\omega^2 + \omega_0^2 - \mathrm{i}\gamma\right) \tilde{Q}_i(\omega) = -\tilde{F}_i \omega^2 \sum_{j=1, j\neq i}^{N} C G_{ij} \tilde{Q}_j(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega/c|\vec{r}_i - \vec{r}_j|}}{|\vec{r}_i - \vec{r}_j|} \tag{4}$$

with

$$G_{ij} = \left[(\hat{r}_{ij} \cdot \hat{z}_i)^2 - 1\right] \left(1 - \frac{3c^2}{\omega^2 |\vec{r}_i - \vec{r}_j|^2} + \frac{3\mathrm{i}c}{\omega|\vec{r}_i - \vec{r}_j|}\right) + 2\left(\frac{\mathrm{i}c}{\omega|\vec{r}_i - \vec{r}_j|} - \frac{c^2}{\omega^2|\vec{r}_i - \vec{r}_j|^2}\right)$$

and

$$\tilde{F}_i = f(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega/c|\vec{r}_i - \vec{r}_j|}}{|\vec{r}_i - \vec{r}_j|}$$

where

$$f(\omega) = \frac{1}{2\pi} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} f(t) \qquad \tilde{Q}(\omega) = \frac{1}{2\pi} \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} Q(t).$$

In the computation, we adopt the following parametrization for equation (4). Define $\tilde{\omega} = \omega/\omega_0$ and $\tilde{\gamma} = \gamma/\omega_0$, and scale all length-based quantities by a and all frequencies by ω_0. In this way, the non-dimensionalized version of equation (4) is written as

$$(1 - \tilde{\omega}^2 - \mathrm{i}\tilde{\omega}\tilde{\gamma})\tilde{Q}_i = -\tilde{f}\frac{\mathrm{e}^{(\mathrm{i}k_0 a)\tilde{k}\tilde{r}_i}}{\tilde{r}_i} - \tilde{C}\sum_{j=1,j\neq i}^{N} G_{ij}\frac{\mathrm{e}^{\mathrm{i}(k_0 a)\tilde{k}|\tilde{\vec{r}}_i - \tilde{\vec{r}}_j|}}{|\tilde{\vec{r}}_i - \tilde{\vec{r}}_j|}\tilde{Q}_j \tag{5}$$

in which

$$\tilde{f} = \frac{fa}{\omega_0^2} \qquad k = \frac{\omega}{c} \qquad k_0 = \frac{\omega_0}{c} \qquad \tilde{k} = \frac{k}{k_0} \qquad \tilde{r} = \frac{r}{a} \qquad \tilde{C} = \tilde{\omega}^2 C/a.$$

The adjustable parameters in equation (5) are $\tilde{C}$, $k_0 a$, and $\tilde{\gamma}$. The source spectrum of the wave transmitted from the source is taken as unity. Equation (5) can be numerically evaluated by a matrix conversion as a function of frequency $\tilde{\omega}$ for different parameter values. After solving for $\tilde{Q}_i$, we write $\tilde{Q}_i$ as

$$\tilde{Q}_i = A_i \mathrm{e}^{\mathrm{i}\theta_i}. \tag{6}$$

Clearly A_i represents the oscillation amplitude, and θ_i denotes the relative oscillation phase of the ith dipole. For each phase, we define a unit phase vector

$$\vec{v}_i = \cos\theta_i\,\vec{e}_x + \sin\theta_i\,\vec{e}_y. \tag{7}$$

We map the phase vectors onto the x–y plane. The starting point of each phase vector is positioned at the centre of its corresponding dipole. Evidently, the phase vectors describe the overall oscillation behaviour of the dipoles, and possible symmetries among these phase vectors would indicate the degree of the coherence of the oscillation behaviour.

Numerical experiments are carried out to study the behaviour of the phase vectors and the site-dependent oscillation amplitudes for all dipoles. The most significant discoveries can be summarized as follows. For sufficiently large coupling constant $\tilde{C}$ and parameter β but small damping rate $\tilde{\gamma}$, the oscillation of the dipoles is persistent for some frequencies slightly above the natural frequency of the dipoles. Within this frequency range, the phase vectors for all dipoles tend to point in the same direction, revealing a surprising coherence of the system. Outside this frequency regime, there is no evident symmetry among the phase vectors. This feature holds for both three and two dimensions and is valid for any random configuration of the dipole clouds at the fixed values of $\tilde{C}$, β, and $\tilde{\gamma}$.

The above results are illustrated by numerical examples. Figure 2 shows the phase vectors for both 2D and 3D cases for an arbitrary random realization of the dipole locations. The number of dipoles tested ranges from 200 to 2500. For better visualization, it is sufficient for us to show the results for $N = 200$. The parameters used in the two particular examples are as follows: (1) 2D: $\beta = 0.5$, $k_0 a = 0.02$, $\tilde{C} = 0.001$, and $\tilde{r} = 0.0001$; (2) 3D: $\beta = 0.1$, $k_0 a = 0.02$, $\tilde{C} = 0.001$, and $\tilde{\gamma} = 0.0001$. The signal generated at the source is assumed to have the phase of zero. It is found that in either low- or high-frequency regimes, the phase vectors for the oscillation of the dipoles point in various directions. There is no apparent ordering in the phase vectors. This is shown by the two top and the two bottom diagrams in figure 2. Within a small regime of frequencies for 2D and a relatively large range of frequencies for 3D, there appears a global ordering in the phase vectors. As shown by the two middle diagrams in figure 2 at $\omega/\omega_0 = 1.326$ for 2D and 1.32 for 3D, all phase vectors point nearly in the negative x-direction. This indicates that all the dipoles tend to oscillate in phase, but in the opposite phase to the source. The phase ordering phenomenon disappears under the following conditions:

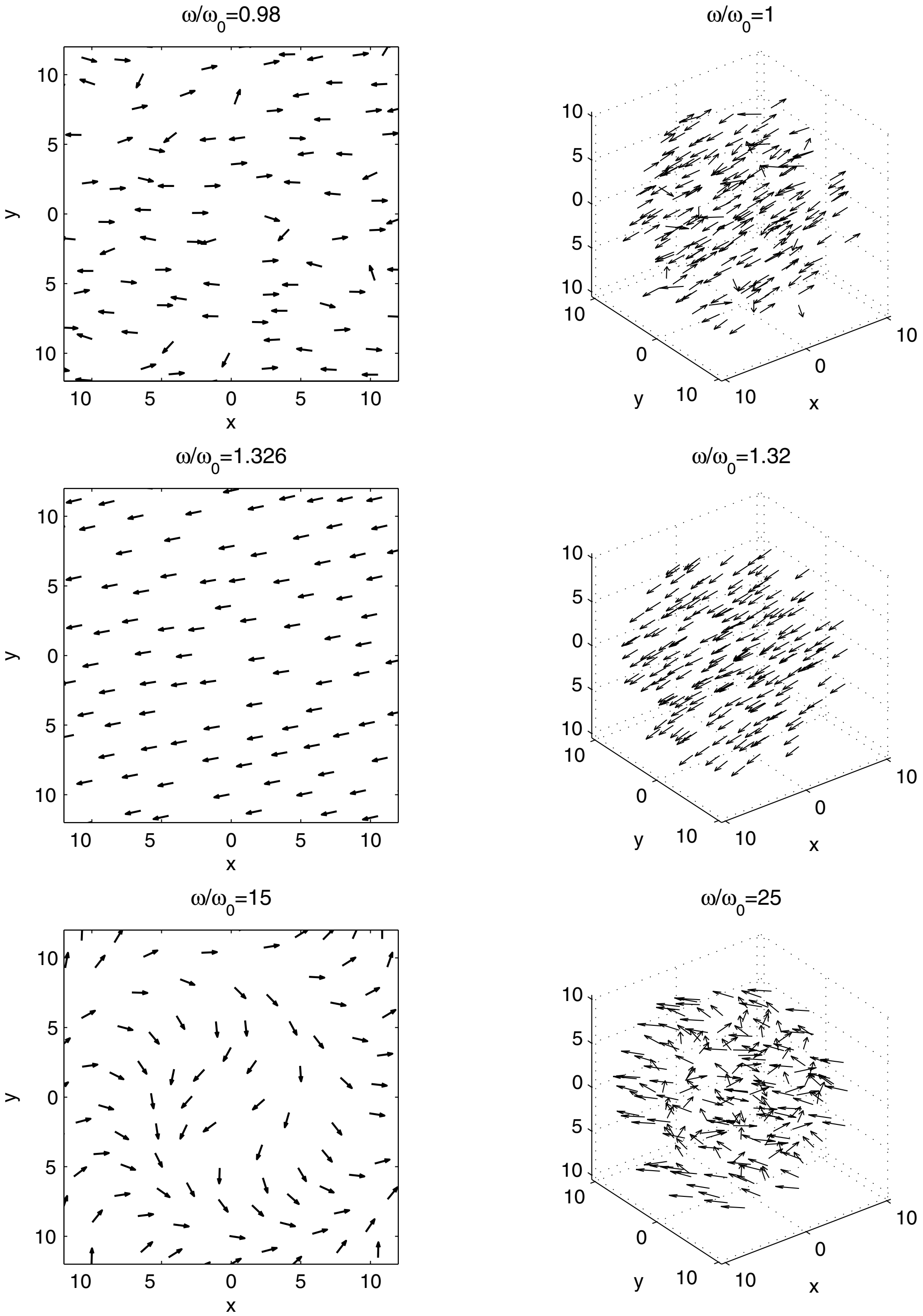

Figure 2. Phase diagrams for the phase vectors for three frequencies. Left column: phase vectors in 2D. Right column: phase vectors in 3D.

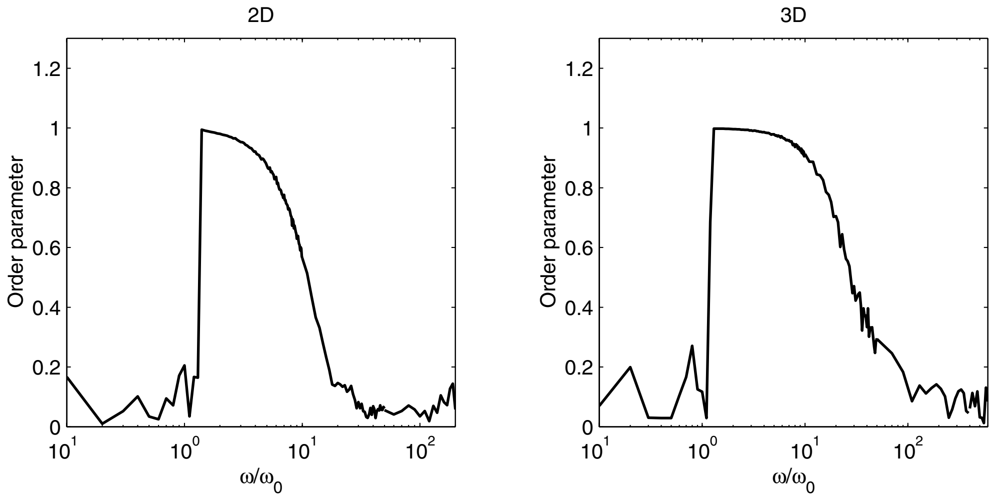

Figure 3. The order parameter as a function of the reduced frequency.

(1) the dissipation exceeds a certain value; this is because sufficient dissipation will break the balance between the energy conversion from radiation to vibration and from vibration to radiation;
(2) the strength of coupling between the dipoles is too small;
(3) the density of the dipoles is not large enough.

Reducing either the dipole interaction or the density of dipoles will reduce the correlation between the dipoles, so long-range ordering cannot be established. We also found that the ordering cannot be induced when the coupling is exceedingly large. A reason may be that when the coupling between dipoles is too big, the dipoles experience mainly the interaction with nearby dipoles, so it is difficult to establish long-range correlation.

In order to explore the transition between the ordered and the random phase states depicted by figure 2, we define an order parameter as follows:

$$\Delta = \frac{1}{N}\left|\sum_{i=1}^{N} \vec{v}_i\right|. \tag{8}$$

It is easy to see that when all dipoles oscillate completely in phase, the order parameter is one. In figure 3, we plot the order parameter as a function of the reduced frequency ω/ω_0 for the two parameter sets in figure 2. It is shown that in both 2D and 3D cases, the order parameter rises to unity rapidly from the low-frequency side, then decreases gradually as the frequency increases. The figure also implies that the phase ordering phenomenon is relatively easy to observe in 3D. The phase ordering range decreases as β decreases. Such a phase ordering phenomenon only appears in a range of frequencies just above the natural frequency of the dipoles.

The phase behaviour described in this paper also shows a similarity with the magnetic phase transition. At low temperatures, the interaction between magnetic dipoles may overcome the thermal fluctuation effect with the result that all magnetic dipoles tend to point in the same direction, showing a magnetic order. When the temperature is increased, the thermal fluctuation becomes more and more dominant and the ordering in the orientation of the magnetic dipoles disappears gradually. In the present case, the main controlling parameter is the frequency of excitation. How do we observe the phase ordering phenomenon described in this paper? In our opinion, the in-phase oscillation would imply a coherent electromagnetic wave inside the system. Therefore, measuring the correlation of the wave at two spatial points is one of the possibilities for discerning the coherent behaviour.

Acknowledgment

The work was supported by the National Science Council.

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